Chapter 3

Laplace Transforms

3.1 Transformation from t to s

Laplace transformation $\mathcal{L}\{.\}$ is used to map time domain functions $f(t)$ into s domain functions $F(s)$. This mapping is$\mathcal{L}\{f(t)\} : f(t) \rightarrow F(s)$ is defined by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt \qquad (3.1)$$

3.1.1 Common Laplace Transforms

Lets find out Laplace transforms of the most frequently encountered time functions: unit impulse $\delta(t)$, unit step $u_s(t)$, time exponent t^n, exponential decay e^{at}, and cosin $cos\omega t$

Unit Impulse, $\delta(t)$

Unit impulse is illustrated in Fig.3.1, and defined as follows

$$\delta(t) = \begin{cases} 0 & \forall t \in (-\infty, 0^-) \\ \int_{0^-}^{+} \delta(t) = 1 & \forall t \in [0^-, 0^+] \\ 0 & \forall t \in (0^+, +\infty) \end{cases} \qquad (3.2)$$

Lets use Laplace transform (3.1) on unit impulse

$$\mathcal{L}\{\delta(t)\} = \int_0^\infty \delta(t)e^{-st}dt \qquad (3.3)$$

However, according to definition (3.2) this integration produces zero value except for the infinitesimal time interval $(0^-, 0^+)$, within which $e^{-st} = e^{-s0} = 1$. Therefore (3.3) simplifies to

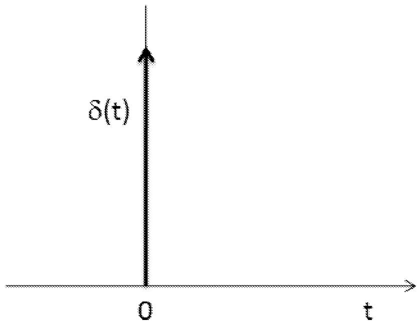


Figure 3.1: Delta function

$$\mathcal{L}\{\delta(t)\} = \int_{0^-}^{0^+} \delta(t).1 \ dt \tag{3.4}$$

According to definition (3.2) the value of this integral is unity. Therefore we have derived the Laplace transform of the unit impulse as

$$\mathcal{L}\{\delta(t)\} = 1 \tag{3.5}$$

Unit Step $u(t)$

Unit step is illustrated in Fig.3.2, and defined as follows

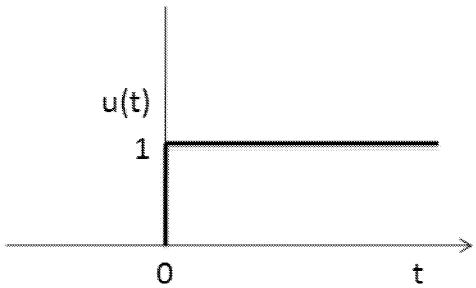


Figure 3.2: Unit step

$$u(t) = \begin{cases} 0 & \forall t \in (-\infty, 0^-) \\ 1 & \forall t \in [0, +\infty) \end{cases} \tag{3.6}$$

Lets use Laplace transform (3.1) on unit step (3.6)

$$\mathcal{L}\{u(t)\} = \int_{0}^{\infty} u(t) e^{-st} dt \tag{3.7}$$

According to definition (3.6), $u_s(t) = 1$ within the limits of integration, therefore

$$\begin{aligned}\mathcal{L}\{u(t)\} &= \int_0^\infty 1.e^{-st}dt \\ &= -\frac{1}{s}e^{-st}|_0^\infty \\ &= \frac{1}{s} \end{aligned} \quad (3.8)$$

Exponential function, e^{at}

According to (3.1), the Laplace transform of an exponential function is

$$\begin{aligned}\mathcal{L}\{e^{at}\} &= \int_0^\infty e^{at}e^{-st}dt \\ &= \int_0^\infty e^{-(s-a)t}dt \\ &= -\frac{1}{s-a}e^{-(s-a)t}|_0^\infty \\ &= \frac{1}{s-a}; \quad (s > a) \end{aligned} \quad (3.9)$$

Cosine Function, $\cos \omega t$

The equivalent expression of $\cos \omega t$

$$\cos \omega t = \frac{1}{2}e^{j\omega t} + \frac{1}{2}e^{-j\omega t} \quad (3.10)$$

is used in (3.1) as follows

$$\begin{aligned}\mathcal{L}\{\cos \omega t\} &= \int_0^\infty \left[\frac{1}{2}e^{j\omega t} + \frac{1}{2}e^{-j\omega t}\right] e^{-st}dt \\ &= \frac{1}{2}\left[\int_0^\infty e^{-(s-j\omega)t}dt + \int_0^\infty e^{-(s+j\omega)t}dt\right] \\ &= \frac{1}{2}\left[-\frac{1}{s-j\omega}e^{-(s-j\omega)t}|_0^\infty - \frac{1}{s+j\omega}e^{-(s+j\omega)t}|_0^\infty\right] \\ &= \frac{1}{2}\left[\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right] \\ &= \frac{s}{s^2+\omega^2} \end{aligned} \quad (3.11)$$

Sine Function, $\sin\omega t$

The equivalent expression for $\sin\omega t$

$$\sin\omega t = \frac{1}{2j}e^{j\omega t} - \frac{1}{2j}e^{-j\omega t} \tag{3.12}$$

is used in (3.1) as follows

$$\begin{aligned}
\mathcal{L}\{\sin\omega t\} &= \int_0^\infty \left[\frac{1}{2j}e^{j\omega t} - \frac{1}{2j}e^{-j\omega t}\right] e^{-st}dt \\
&= \frac{1}{2j}\left[\int_0^\infty e^{-(s-j\omega)t}dt - \int_0^\infty e^{-(s+j\omega)t}dt\right] \\
&= \frac{1}{2j}\left[-\frac{1}{s-j\omega}e^{-(s-j\omega)t}|_0^\infty + \frac{1}{s+j\omega}e^{-(s+j\omega)t}|_0^\infty\right] \\
&= \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right] \\
&= \frac{\omega}{s^2+\omega^2} \qquad (3.13)
\end{aligned}$$

n^{th} **Power of time,** t^n

From (3.1), Laplace transform of t^n is

$$\mathcal{L}\{t^n\} = \int_0^\infty t^n e^{-st}dt \tag{3.14}$$

The RHS can be expanded using following equivalence

$$\int_0^\infty udv = [uv]|_0^\infty - \int_0^\infty vdu \tag{3.15}$$

where

$$\begin{array}{r|l} u = t^n & dv = e^{-st}dt \\ du = nt^{n-1} & v = -\frac{1}{s}e^{-st} \end{array}$$

By substitution of these expressions on (3.14)

$$\begin{aligned}
L\{t^n\} &= [uv]|_0^\infty - \int_0^\infty vdu \qquad (3.16) \\
&= \frac{-t^n e^{-st}}{s}|_0^\infty + \int_0^\infty \frac{1}{s}e^{-st}nt^{n-1}dt \\
&= 0 + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt \\
&= \frac{n}{s}L\left\{t^{n-1}\right\}
\end{aligned}$$

By repetitive transformation we get into the following expression

$$L\{t^n\} = \frac{n!}{s^{n+1}} \tag{3.17}$$

3.1.2 Properties of Laplace Transforms

Linearity

Linearity requires following two conditions to be satisfied: (1) superposition and (2) multiplication by a constant. For two time functions $f_1(t)$ and $f_2(t)$, and two coefficients α_1 and α_2, lets derive Laplace transform of the function $\alpha_1 f_1(t) \pm \alpha_2 f_2(t)$ using definition in (3.1).

$$\begin{aligned}
\mathcal{L}\{\alpha_1 f_1(t) \pm \alpha_2 f_2(t)\} &= \int_0^\infty \{\alpha_1 f_1(t) \pm \alpha_2 f_2(t)\} e^{-st} dt \\
&= \int_0^\infty \alpha_1 f_1(t) e^{-st} dt \pm \int_0^\infty \alpha_2 f_2(t) e^{-st} dt \\
&= \alpha_1 \int_0^\infty f_1(t) e^{-st} dt \pm \alpha_2 \int_0^\infty f_2(t) e^{-st} dt \\
&= \alpha_1 \mathcal{L}\{f_1(t)\} \pm \alpha_2 \mathcal{L}\{f_2(t)\} \qquad (3.18)
\end{aligned}$$

which confirms that Laplace transform is linear.

Exponential Scaling

Exponential scaling of a function $f(t)$ is described by $e^{at} f(t)$. Using (3.1) its Laplase transform can be obtained as follows

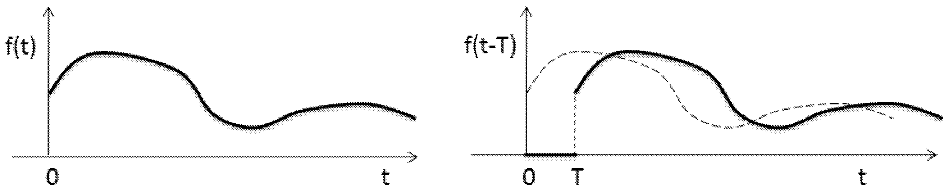
$$\begin{aligned}
\mathcal{L}\{e^{at} f(t)\} &= \int_0^\infty e^{at} f(t) e^{-st} dt \\
&= \int_0^\infty e^{-(s-a)t} f(t) dt \\
&= F(s-a) \qquad (3.19)
\end{aligned}$$

Example: Determine the Laplace transform of $e^{-2t} \cos 3t$
Answer: This is an exponential scaling e^{-2t} of $\cos 3t$. From (3.11), with ω=3, the Laplace transform $L\{\cos 3t\} = s/(s^2 + 3^2)$. When this is exponentially scaled with e^{-2t}, the resulting Laplace transform is given by (3.19) in that s is replaced with $s - (-2) = s + 2$. Therefore,

$$\mathcal{L}\{e^{-2t} \cos 3t\} = \frac{s+2}{(s+2)^2 + 3^2} = \frac{s+2}{s^2 + 4s + 13} \tag{3.20}$$

Time Delay

The time delayed function $f(t-T)$ by time T is illustrated in Fig. (3.3), and defined as



(a) (b)

Figure 3.3: (a) $f(t)$ (b) its time delayed version $f(t-T)$

$$f(t-T) = \begin{cases} 0 & 0 \leq t < T \\ f(t-T) & T \leq t \end{cases} \tag{3.21}$$

According to (3.1), the Laplace transform of $f(t)$ is

$$\mathcal{L}\{f(t-T)\} = \int_0^\infty f(t-T)e^{-st}dt \tag{3.22}$$

Lets introduce new time variable $\tau = t - T$. This new variable must replace t-dependent expressions; dt,$\int_0^\infty$, and e^{-st} in (3.22). Following expressions (1) $t = \tau + T$, (2) $dt = d\tau$, (3) when $t \to 0$, then $\tau \to -T$, and (4) when $t \to \infty$ then $\tau \to \infty$ can be used to rewrite the right hand side of (3.22) with τ as the time variable

$$\begin{aligned} \mathcal{L}\{f(t-T)\} &= \int_{-T}^\infty f(\tau)^{-s(\tau+T)}d\tau \\ &= \int_{-T}^0 f(\tau)^{-s(\tau+T)}d\tau + \int_0^\infty f(\tau)^{-s(t+T)}d\tau \end{aligned} \tag{3.23}$$

The first integral on the RHS has to be dropped because Laplace definition does not consider negative time. Therefore,

$$\begin{aligned} \mathcal{L}\{f(t-T)\} &= \int_0^\infty f(\tau)^{-s(\tau+T)}d\tau \\ &= e^{-sT}\int_0^\infty f(\tau)^{-s\tau}d\tau \\ &= e^{-sT}F(s) \end{aligned} \tag{3.24}$$

Example: Derive the Laplace transform of the unit pulse $p_{2,5}$ going positive within the period $t \in [2, 5]$.

Answer: We don't have to derived Laplace transform for pulse waveforms from the first principles, because we can use time-shifted step functions to generate pulse waveforms. In fact, if we use $u(t-2)$, which is the delayed step occur at $t = 2$, and subtract from it $u(t-5)$, which is the delayed step that appear at $t = 5$, then what we have is the $p_{2,5}$ pulse going positive from $t = 2$ to $t = 5$ as shown in Fig.3.4

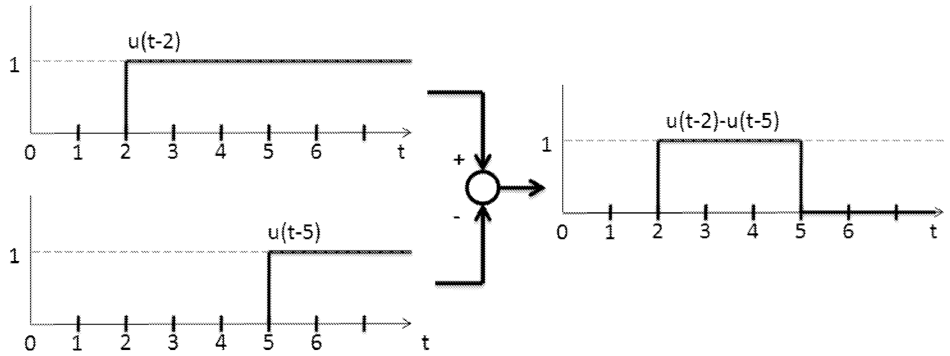


Figure 3.4: Superposition of two step functions to generate a pulse

We can write

$$p_{2,5} = u(t-2) - u(t-5) \tag{3.25}$$

and use (3.1) on (3.25) to write

$$\mathcal{L}\{p_{2,5}\} = L\{u(t-2)\} - L\{u(t-5)\} \tag{3.26}$$

using (3.24)

$$\begin{aligned} \mathcal{L}\{p_{2,5}\} &= e^{-2s} L\{u_s(t)\} - e^{-5s} L\{u_s(t)\} \\ &= e^{-2s}\frac{1}{s} - e^{-5s}\frac{1}{s} \\ &= \frac{e^{-2s} - e^{-5s}}{s} \end{aligned} \tag{3.27}$$

Multiplication by time, $tf(t)$

From (3.1), the Laplace transform of a time multiplied function $tf(t)$ is

$$\begin{aligned}\mathcal{L}\{tf(t)\} &= \int_0^\infty tf(t)e^{-st}dt \\ &= \int_0^\infty f(t)\frac{d}{ds}\{-e^{-st}\}dt \\ &= -\frac{d}{ds}\left\{\int_0^\infty f(t)e^{-st}dt\right\} \\ &= -\frac{d}{ds}F(s) \end{aligned} \tag{3.28}$$

In general, it can be shown that

$$\mathcal{L}\{t^k f(t)\} = (-1)^k \frac{d^k}{ds^k}F(s) \tag{3.29}$$

Laplace Transform of a Function Derivative

Lets use (3.1) on function derivative $\frac{f(t)}{dt}$ as follows

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = \int_0^\infty \frac{df(t)}{dt}e^{-st}dt \tag{3.30}$$

Using the method of integration by parts used in (3.17), lets assume following substitutions as follows

$u = e^{-st}$	$dv = \frac{df(t)}{dt}dt$
$du = -se^{-st}dt$	$v = f(t)$

Then, (3.30) can be written using equivalence $\int_0^\infty udv = uv|_0^\infty - \int_0^\infty vdu$ as follows.

$$\begin{aligned}\mathcal{L}\left\{\frac{df(t)}{dt}\right\} &= f(t)e^{-st}|_0^\infty - \int_0^\infty -f(t)se^{-st}dt \\ &= -f(0) + s\int_0^\infty f(t)e^{-st}dt \\ &= sF(s) - f(0)\end{aligned} \tag{3.31}$$

In general, the Laplace transform of the n^{th} order derivative $\frac{d^n f(t)}{dt^n}$ is

$$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{n-1}(0) \tag{3.32}$$

Laplace Transform of a Function Integral

Lets use (3.1) on function integral $\int f(\tau)d\tau$ as follows

$$\mathcal{L}\left\{\frac{df(\tau)}{d\tau}\right\} = \int_{t=0}^{\infty}\left\{\int_{\tau=0}^{t} f(\tau)d\tau\right\} e^{-st}dt \tag{3.33}$$

The double integration here is as shown in Fig. 3.5(a). However, the order of double integration can be changed as shown in Fig. 3.5(b) that traverses the same surface of integration.

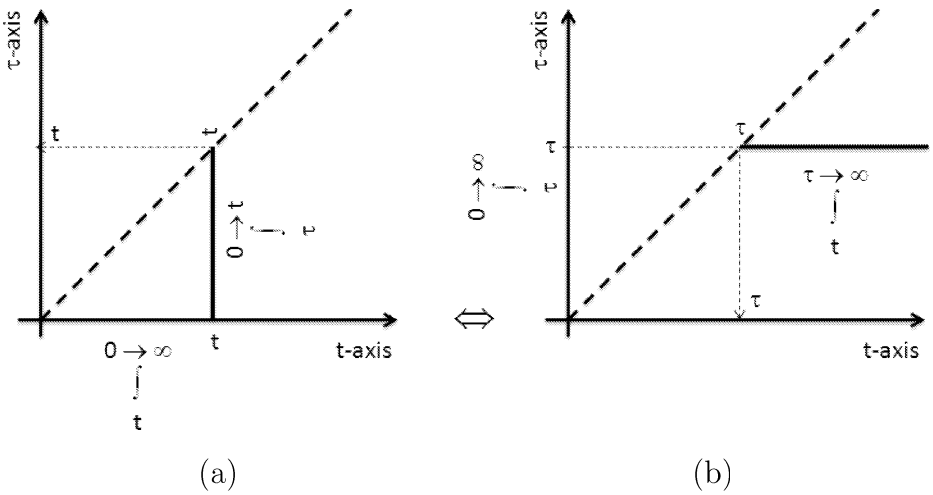


Figure 3.5: Double integration: order and corresponding limits

Therefore, we can write (3.33) as follows.

$$\begin{aligned}
&= \int_{\tau=0}^{\infty}\int_{t=\tau}^{\infty} f(\tau)e^{-st}dtd\tau \\
&= \int_{\tau=0}^{\infty} f(\tau)\left\{\int_{t=\tau}^{\infty} e^{-st}dt\right\} d\tau \\
&= \int_{\tau=0}^{\infty} f(\tau)\left\{-\frac{1}{s}e^{-st}|_{\tau}^{\infty}\right\} d\tau \\
&= \frac{1}{s}\int_{\tau=0}^{\infty} f(\tau)e^{-s\tau}d\tau \\
&= \frac{1}{s}F(s)
\end{aligned} \tag{3.34}$$

Time Scaling, $f(at)$

Time scaled function $f(at)$ is a compressed, or stretched version of $f(t)$ by the scale factor a. If $a < 1$ function compresses (frequency increases), and if

$a > 1$ function stretches (frequency drops). Lets use (3.1) on a time-scaled function as follows

$$\mathcal{L}\{f(at)\} = \int_0^\infty f(at)e^{-st}dt \tag{3.35}$$

Lets introduce new time variable $\tau = at$, and replace with it all t terms in (3.35). We can say that $\frac{1}{a}dt = d\tau$, and when $t \to 0, \tau \to 0$, and when $t \to \infty, \tau \to \infty$. Then, we have

$$\begin{aligned} L\{f(at)\} &= \frac{1}{a}\int_0^\infty f(\tau)e^{-\frac{s\tau}{a}}d\tau \\ &= \frac{1}{a}\int_0^\infty f(\tau)e^{-\frac{s}{a}\tau}d\tau \\ &= \frac{1}{a}F(s/a) \end{aligned} \tag{3.36}$$

Initial Value Theorem

An interesting piece of knowledge can be discovered if we evaluate the limit of (3.31) when $s \to \infty$ as follows

$$lim_{s\to\infty}\left\{\int_0^\infty \frac{df(t)}{dt}e^{-st}dt\right\} = lim_{s\to\infty}\left\{sF(s) - f(0)\right\} \tag{3.37}$$

The LHS of this expression has e^{-st} which has zero value as $s \to \infty$. And, on the RHS, $f(0)$ is independent of variable s. Therefore, we can write (3.37) as follows

$$0 = lim_{s\to\infty}sF(s) - f(0) \tag{3.38}$$

In fact, $f(0)$ can be written alternatively as the value of $f(t)$ as $t \to 0$. Therefore, we have

$$lim_{t\to 0}f(t) = lim_{s\to\infty}sF(s) \tag{3.39}$$

which tells us that if you have $F(s)$, and you want to know $f(0)$ all what you need to do is to determine the limit of $F(s)$ as $s \to \infty$. This way, there is no need of transforming $F(s)$ back to $f(t)$ to find out the initial value.

Final Value Theorem

Another very useful piece of knowledge can be discovered if we evaluate the limit of (3.31) when $s \to 0$ as follows.

$$lim_{s\to 0}\left\{\int_0^\infty \frac{df(t)}{dt}e^{-st}dt\right\} = lim_{s\to 0}\left\{sF(s) - f(0)\right\} \tag{3.40}$$

As $e^{-st} = 1$ when $s = 0$, and also $f(0)$ is independent of s, we can write (3.40) as follows

$$\begin{aligned}\int_0^\infty \frac{df(t)}{dt}.1dt &= lim_{s\to 0}\left\{sF(s)\right\} - f(0) \\ f(t)|_0^\infty &= lim_{s\to 0}\left\{sF(s)\right\} - f(0) \\ f(\infty) - f(0) &= lim_{s\to 0}\left\{sF(s)\right\} - f(0) \\ f(\infty) &= lim_{s\to 0}\left\{sF(s)\right\}\end{aligned} \tag{3.41}$$

In fact, $f(\infty)$ can be written alternatively as the value of $f(t)$ as $t \to \infty$. Therefore, we have

$$lim_{t\to\infty}f(t) = lim_{s\to 0}sF(s) \tag{3.42}$$

which tells us that if you have $F(s)$, and you want to know $f(\infty)$ all what you need to do is to set $s = 0$ in $sF(s)$. This way, there is no need of transforming $F(s)$ back to $f(t)$ to determine the final value of the function.

Convolution Integral $f(t) * g(t)$, the Plant Response

The convolution integral of two time functions $r(t)$ and $g(t)$ is defined as.

$$r(t) * g(t) = \int_0^t r(\tau)g(t-\tau)d\tau \tag{3.43}$$

which is illustrated in Fig.3.6

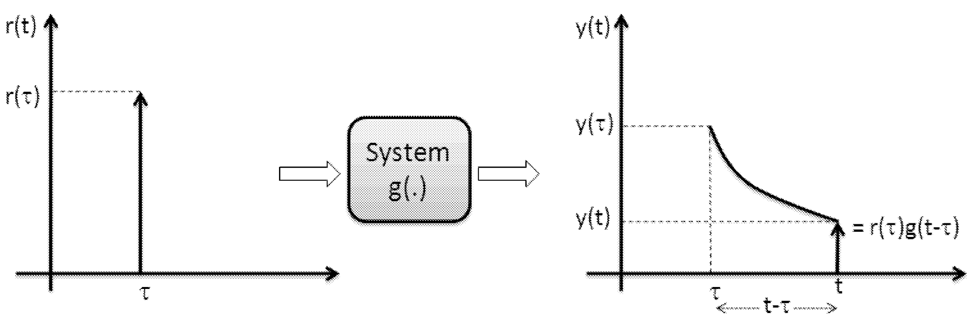


Figure 3.6: Exogenous input $r(\tau)$ and its response $y(t)$ after elapsed time $t - \tau$

Here, $r(\tau)$ (the instantaneous value of the $r(t)$) is an impulsive excitation at time=τ to the system. The system immediately produces a response $y(\tau)$, which, after time interval of $t-\tau$ attains a value of $y(t)=r(\tau)g(t-\tau)$, where $g(.)$ is the unit impulse response of the system. If excitation is a continuous waveform $r(t); 0 \leq t \leq \infty$, then total response at time t should be the integral of the responses of all impulses $r(t); 0 \leq t \leq t$, which is the convolution integral in (3.43). The Laplace transform of a convolution integral can be determined by operating (3.1) on (3.43) as follows.

$$\mathcal{L}\{r(t)*g(t)\} = \int_0^\infty \left\{\int_0^t r(\tau)g(t-\tau)d\tau\right\} e^{-st}dt \tag{3.44}$$

By changing the order of double integration as explained in Fig.3.5, we can write

$$\mathcal{L}\{r(t)*g(t)\} = \int_0^\infty \int_0^\infty r(\tau)g(t-\tau)e^{-st}dtd\tau \tag{3.45}$$

Lets introduce new time variable u as follows.

$$\begin{array}{l|l} u = t-\tau & t \to \tau, u \to 0 \\ du = dt & t \to \infty, u \to \infty \end{array}$$

Then we can write (3.45) as follows.

$$\begin{aligned} \mathcal{L}\{r(t)*g(t)\} &= \int_0^\infty \int_0^\infty r(\tau)g(u)e^{-s(\tau+u)}dud\tau \\ Y(s) &= \int_0^\infty r(\tau)e^{-s\tau}d\tau \int_0^\infty g(u)e^{-su}du \\ Y(s) &= R(s)G(s) \end{aligned} \tag{3.46}$$

This shows that the Laplace transform of the system's response $Y(s)$ is given by the product of Laplace transform of excitation $R(s)$ and Laplace transform of system's impulse transfer function, which we name as the transfer function of the system $G(s)$.

3.2 Response using Laplace Transforms

3.2.1 First Order System (RC Circuit)

Lets look at how we could use Laplace transforms to solve system model and obtain system response efficiently. To demonstrate this, we go back to the model (2.3) in chapter 2.1 and try to apply Laplace techniques to solve it for system response. Assuming a step supply voltage $v_s(t) = V_s u_s(t)$, lets take Laplace transform of both sides using (3.31) as follows

$$\begin{aligned} sV(s) - v(0) + aV(s) &= bV_s\frac{1}{s} \\ (s+a)V(s) &= v(0) + bV_s\frac{1}{s} \\ V(s) &= \frac{1}{s+a}v(0) + \frac{b}{s(s+a)}V_s \end{aligned} \tag{3.47}$$

We could evaluate inverse Laplace transform of (3.47) in two ways using partial fractions or convolution integral. Lets workout the two approaches as follows.

Partial Fractions Method:
Lets take partial fractions of (3.47) as follows

$$V(s) = \frac{1}{s+a}v(0) + \frac{b}{a}\left(\frac{1}{s} - \frac{1}{s+a}\right)V_s \tag{3.48}$$

of which inverse Laplace transform is

$$v(t) = v(0)e^{-at} + \frac{b}{a}V_s\left(1 - e^{-at}\right) \tag{3.49}$$

which is the same response that has been obtained earlier in (2.12).

Convolution Integral Method:
We can identify the second term of (3.47) as a product of two Laplace transforms $\frac{1}{s}$ and $\frac{1}{s+a}$ for which the inverse Laplace transform is the convolution integral of the two inverse Laplace transforms $u_s(t)$ and e^{-at}. Therefore, the total response can be written as follows.

$$\begin{aligned} v(t) &= v(0)e^{-at} + bV_s\int_0^t e^{-a\tau}u_s(t-\tau)d\tau \\ &= v(0)e^{-at} + bV_s\int_0^t e^{-a\tau}.1d\tau \\ &= v(0)e^{-at} + \frac{-b}{a}V_s e^{-a\tau}|_0^t \\ &= v(0)e^{-at} + \frac{b}{a}V_s\left(1 - e^{-at}\right) \end{aligned} \tag{3.50}$$

which is the same response that has been obtained earlier in (2.12).

3.2.2 Simulation

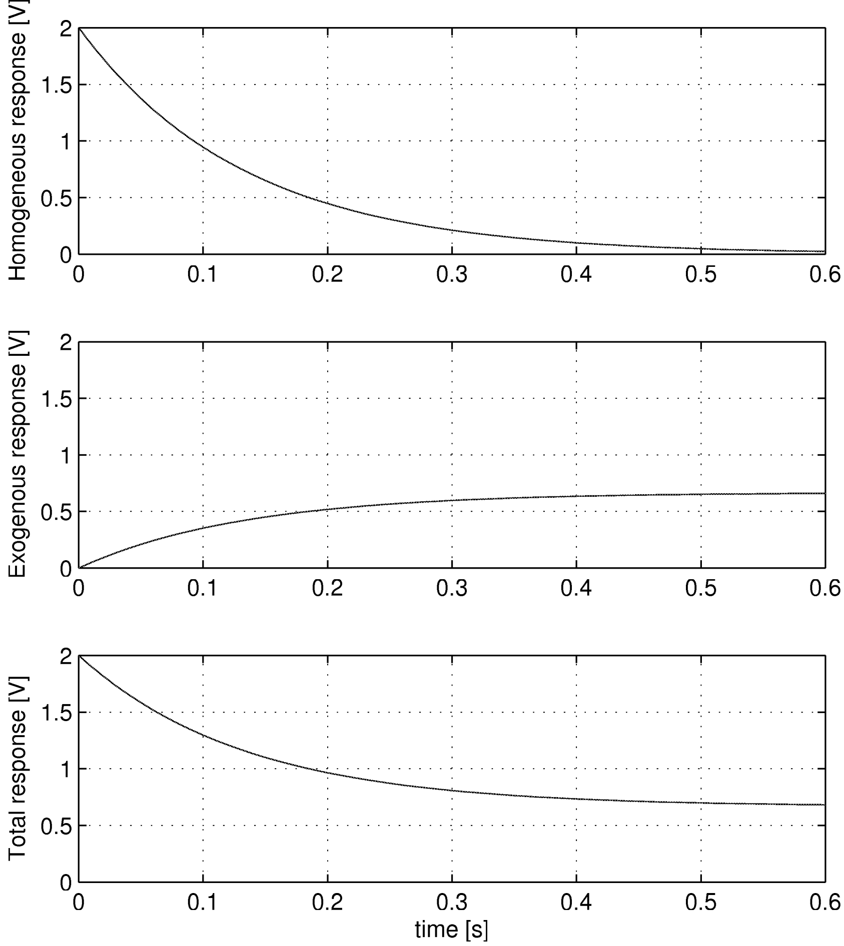


Figure 3.7: RC circuit response

For $R_1 = R_2$=1$k\Omega$, and C=200μF, and assuming that there is a voltage across the capacitor initially $v(0)$=2[V], the homogeneous, exogenous, and total responses are illustrated in Fig.3.7.

3.2.3 Second Order System: shock-absorber

Lets look at the second order shock-absorber dynamics in (2.2), and re-write it as follows

$$\ddot{y}(t) + 2\sigma\dot{y}(t) + \rho y(t) = \eta f(t) \tag{3.51}$$

where $2\sigma = \frac{b}{m}$, $\rho = \frac{k}{m}$, and $\eta = \frac{1}{m}$. Using Laplace transforms of a derivative (3.32), we can transform above equation into Laplace domain as follows

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + 2\sigma[sY(s) - y(0)] + \rho Y(s) &= \eta F(s) \\ (s^2 + 2\sigma s + \rho)Y(s) - y(0)s - [2\sigma y(0) + y'(0)] &= \eta F(s) \end{aligned}$$

$$Y(s) = \frac{y(0)s + [2\sigma y(0) + y'(0)]}{(s^2 + 2\sigma s + \rho)} + \frac{\eta}{(s^2 + 2\sigma s + \rho)}F(s) \tag{3.52}$$

Using the denominator expression, we generate the characteristic equation of the system $\Delta(s) = s^2 + 2\sigma s + \rho = 0$ of which the roots dictate system's characteristics (therefore called the characteristic equation). The solutions, or roots of the characteristic equation are also called the poles of the system. Due to the quadratic nature of the characteristic equation, there are three possibilities for the nature of the roots: (1) **real distinct**, (2) **real coincident**, and (3) **complex**. Lets analyze how the system response is dictated by poles in these three possibilities.

Case 1 $\sigma^2 - \rho > 0$ $(b > 2\sqrt{mk}) \Rightarrow$ **real, negative, distinct poles:**

Lets introduce $K_1 = y(0)$, $K_2 = 2\sigma y(0) + y'(0)$. The two distinct, real poles α_1 and α_2 are

$$\alpha_1, \alpha_2 = -\sigma \pm \sqrt{\sigma^2 - \rho} \tag{3.53}$$

Then, (3.52) can be written as

$$Y(s) = \frac{K_1 s + K_2}{(s - \alpha_1)(s - \alpha_2)} + \frac{\eta}{(s - \alpha_1)(s - \alpha_2)}F(s) \tag{3.54}$$

which, can be expanded using partial fractions as follows

$$\begin{aligned} Y(s) &= \left\{ \frac{P_1}{(s - \alpha_1)} + \frac{P_2}{(s - \alpha_2)} \right\} + \eta \left\{ \frac{P_3}{(s - \alpha_1)} + \frac{P_4}{(s - \alpha_2)} \right\} F(s) \\ &= \frac{P_1}{(s - \alpha_1)} + \frac{P_2}{(s - \alpha_2)} + \eta \frac{P_3}{(s - \alpha_1)}F(s) + \eta \frac{P_4}{(s - \alpha_2)}F(s) \end{aligned} \tag{3.55}$$

where the residues $P_1 = \frac{\alpha_1 K_1 + K_2}{\alpha_1 - \alpha_2}$, $P_2 = \frac{\alpha_2 K_1 + K_2}{\alpha_2 - \alpha_1}$, $P_3 = \frac{1}{\alpha_1 - \alpha_2}$, and $P_4 = \frac{1}{\alpha_2 - \alpha_1}$ can be determined by partial fraction cover up method, which is demonstrated below for the determination of P_1 only.

$$
\begin{aligned}
lim_{s\to\alpha_1}(s-\alpha_1)\frac{K_1 s+K_2}{(s-\alpha_1)(s-\alpha_2)} &= lim_{s\to\alpha_1}(s-\alpha_1) \\
&\quad \left\{\frac{P_1}{(s-\alpha_1)}+\frac{P_2}{(s-\alpha_2)}\right\} \\
lim_{s\to\alpha_1}\frac{K_1 s+K_2}{(s-\alpha_2)} &= lim_{s\to\alpha_1}\left\{P_1+(s-\alpha_1)\frac{P_2}{(s-\alpha_2)}\right\} \\
\frac{K_1\alpha_1+K_2}{\alpha_1-\alpha_2} &= P_1
\end{aligned}
$$

The cover up method can be generalized as follows for an expression $\frac{K(s)}{(s-\alpha_1)(s-\alpha_1)\cdots(s-\alpha_i)\cdots(s-\alpha_n)}$ as follows

$$
P_i = lim_{s\to\alpha_i}(s-\alpha_i)\left\{\frac{K(\alpha_i)}{(s-\alpha_1)(s-\alpha_1)\cdots(s-\alpha_i)\cdots(s-\alpha_n)}\right\} \qquad (3.56)
$$

Convolution Integral Method: We can clearly notice in (3.55) that the first two terms on the RHS are Laplace transforms of exponential functions as derived in (3.9), whereas each of the last two terms is a product of two Laplace transforms, which becomes a convolution integral in time domain as shown in (3.45). Therefore, the total solution is given by

$$
y(t) = P_1 e^{\alpha_1 t}+P_2 e^{\alpha_2 t}+\eta P_3\int_0^t e^{\alpha_1(t-\tau)}f(\tau)d\tau+\eta P_4\int_0^t e^{\alpha_2(t-\tau)}f(\tau)d\tau \quad (3.57)
$$

which can be solved for known forcing functions $f(\tau)$. Any function can be synthesized by superimposing a set of pulse waveforms of specific set-in times and amplitude, therefore, lets determine the response in (3.57) for a step function $f(\tau) = Au_s(\tau)$. As $Au_s(\tau) = A; 0 \leq \tau$, (3.57) can be written as

$$
\begin{aligned}
y(t) &= P_1 e^{\alpha_1 t}+P_2 e^{\alpha_2 t}+\eta P_3\int_0^t e^{\alpha_1(t-\tau)}Ad\tau+\eta P_4\int_0^t e^{\alpha_2(t-\tau)}Ad\tau \\
&= P_1 e^{\alpha_1 t}+P_2 e^{\alpha_2 t}+\eta P_3 e^{\alpha_1 t}\int_0^t e^{-\alpha_1\tau}Ad\tau+\eta P_4 e^{\alpha_2 t}\int_0^t e^{-\alpha_2\tau}Ad\tau \\
&= P_1 e^{\alpha_1 t}+P_2 e^{\alpha_2 t}-\frac{\eta P_3 A e^{\alpha_1 t}}{\alpha_1}e^{-\alpha_1\tau}|_0^t-\frac{\eta P_4 A e^{\alpha_2 t}}{\alpha_2}e^{-\alpha_2\tau}|_0^t
\end{aligned}
$$

$$
\begin{aligned}
&= P_1e^{\alpha_1 t} + P_2e^{\alpha_2 t} - \frac{\eta P_3 A e^{\alpha_1 t}}{\alpha_1}(e^{-\alpha_1 t} - 1) - \frac{\eta P_4 e^{\alpha_2 t} A}{\alpha_2}(e^{-\alpha_2 t} - 1) \\
&= P_1e^{\alpha_1 t} + P_2e^{\alpha_2 t} - \frac{\eta P_3 A}{\alpha_1}(1 - e^{\alpha_1 t}) - \frac{\eta P_4 A}{\alpha_2}(1 - e^{\alpha_2 t}) \\
&= -\eta A\left(\frac{P_3}{\alpha_1} + \frac{P_4}{\alpha_2}\right) + \left(P_1 + \frac{\eta A P_3}{\alpha_1}\right)e^{\alpha_1 t} + \left(P_2 + \frac{\eta A P_4}{\alpha_2}\right)e^{\alpha_2 t}
\end{aligned}
\tag{3.58}
$$

Here, the first term on the RHS is time-independent, and is the *steady state response*, whereas the last two terms are time-decaying, and together form the *transient response*. Moreover, the first term depends entirely on P_3 and P_4 telling us that the steady state response depends entirely on the forcing function. It is also interesting to notice that the last two terms depend on all four residues P_1, P_2, P_3, and P_4 indicating that the transient response is contributed by the initial conditions of the response as well as the forcing function.

If we are interested in knowing what will eventually happen to the response only steady state response is needed to be determined. On the other hand, if we are interested in knowing how the shock-absorber squeezes, and how long it will take to become steady, then transient response has to be determined.

Partial Fraction Method: If you find it difficult to work through convolution integral you can still solve (3.54) algebraically in Laplace domain using partial fractioning method. In fact, Laplace transforms help solving differential equations while avoiding integration and differentiation. For the step function $f(t) = Au_s(t)$ the Laplace transform is $F(s) = \frac{A}{s}$, as derived in (3.8). Then we can write (3.54) as follows

$$
\begin{aligned}
Y(s) &= \frac{K_1 s + K_2}{(s-\alpha_1)(s-\alpha_2)} + \frac{\eta}{(s-\alpha_1)(s-\alpha_2)}\frac{A}{s} \qquad (3.59)\\
&= \frac{P_1}{(s-\alpha_1)} + \frac{P_2}{(s-\alpha_2)} + \eta A\left(\frac{Q_1}{(s-\alpha_1)} + \frac{Q_2}{(s-\alpha_2)} + \frac{Q_3}{s}\right)
\end{aligned}
$$

where $Q_1 = \frac{1}{\alpha_1(\alpha_1-\alpha_2)}, Q_2 = \frac{1}{\alpha_2(\alpha_2-\alpha_1)}, Q_3 = \frac{1}{\alpha_1\alpha_2}$ can be determined using the cover up method demonstrated in (3.56). It is interesting to see that all terms on the RHS are familiar Laplace transforms. The first four terms came from exponential functions, whereas the last term is the Laplace transform of a step function. Therefore, the total response is given by

$$
y(t) = P_1e^{\alpha_1 t} + P_2e^{\alpha_2 t} + \eta A\left(Q_1e^{\alpha_1 t} + Q_2e^{\alpha_2 t}\right) + \eta A Q_3
$$

$$= \eta A Q_3 + (P_1 + \eta A Q_1)e^{\alpha_1 t} + (P_2 + \eta A Q_2)e^{\alpha_2 t} \qquad (3.60)$$

which is in fact the same response in (3.58), which was obtained using convolution integral. Now that we have determined the total response using convolution integralmethod and Laplace transform nethod, we can conclude that the algebraic method of solving ODEs using Laplace transforms is very convenient and effective.

3.2.4 Simulation

Lets assume the shock-absorber parameters as follows: damping stiffness k=125[N/cm], which means that a 125N force is required to squeeze the spring by 1[cm], and damping constant b=700 [Ns/cm], which means that 1[cm/s] speed is opposed by a 700[N] force. Then, from (3.51) σ=7, ρ=2.5, and η=0.2. We have in this case real distinct pair of poles of the characteristic equation as α_1=-0.181 and α_2=-13.819. This combination of stronger damping over spring stiffness is known as *over damped*, in which case motion is overly suppressed by the damper. Lets also assume some non-zero initial conditions, that is, at t=0, the shock absorber was not at rest but was squeezing at a position $y(0)$=-1.5[cm] with a speed $y'(0)$=-1.8[cm/s]. The homogeneous, exogenous, and total responses of the shock-absorber are illustrated in Fig.3.8.

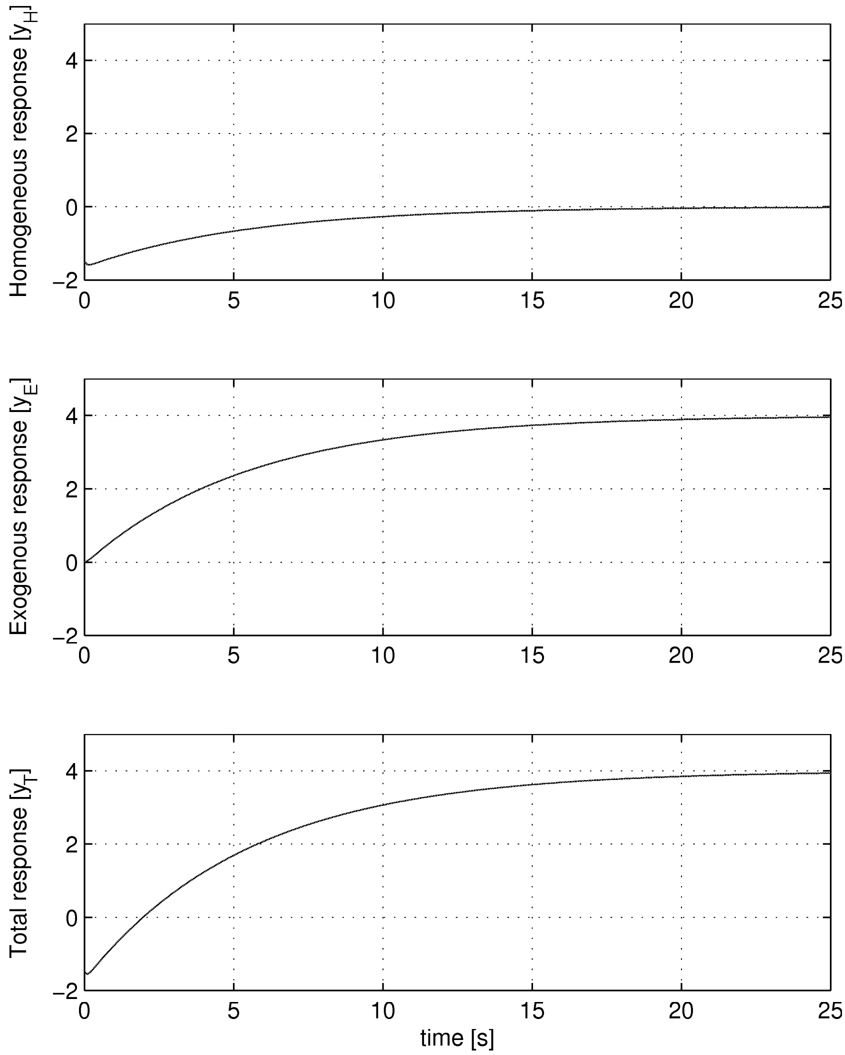


Figure 3.8: Shock absorber squeezing with the 50kg person getting on board

Case 2 $\sigma^2 - \rho = 0$**,** $(b = 2\sqrt{mk}) \Rightarrow$ **real, negative coincident poles:**

In this case, the two roots are $\alpha_1 = \alpha_2 = \alpha = -\sigma$. Then, (3.52) can be written as

$$Y(s) = \frac{K_1 s + K_2}{(s-\alpha)^2} + \frac{\eta}{(s-\alpha)^2}F(s) \tag{3.61}$$

Using partial fraction, this can be resolved into

$$Y(s) = \left\{\frac{P_5 s}{(s-\alpha)^2} + \frac{P_6}{(s-\alpha)}\right\} + \eta\left\{\frac{P_7 s}{(s-\alpha)^2} + \frac{P_8}{(s-\alpha)}\right\}F(s) \tag{3.62}$$

where the residues $P_5 = \frac{K_1\alpha + K_2}{\alpha}$, $P_6 = -\frac{K_2}{\alpha}$, $P_7 = \frac{1}{\alpha}$, and $P_8 = -\frac{1}{\alpha}$ can be determined by equating coefficients. Again, we recognize within the first set of curly braces the homogeneous response, whereas the exogenous response can be recognized within the second set of culy braces. We can use $f(t) = Au_s(t)$ whose Laplace transform is $F(s) = A\frac{1}{s}$ and determine the response as follows

$$\begin{aligned} Y(s) &= \left\{\frac{P_5 s}{(s-\alpha)^2} + \frac{P_6}{(s-\alpha)}\right\} + \eta\left\{\frac{P_7 s}{(s-\alpha)^2} + \frac{P_8}{(s-\alpha)}\right\}A\frac{1}{s} \\ &= P_5\frac{s}{(s-\alpha)^2} + P_6\frac{1}{(s-\alpha)} + \eta A P_7\frac{1}{(s-\alpha)^2} + \eta A P_8\frac{1}{s(s-\alpha)} \end{aligned} \tag{3.63}$$

Here, the second term on the RHS is familiar to us as the Laplace transform of $e^{\alpha t}$ as derived in (3.9). And, the third term is the derivative of the second term with respect to s, therefore, in time domain third term appears as $te^{\alpha t}$ according to (3.28). The first term is identical to third term except for that s in the numerator, which tells us that in time domain it should be $\frac{d(te^{\alpha t})}{dt} + te^{\alpha t}|_{t=0} = (1+\alpha t)e^{\alpha t}$ as we have derived in (3.31). Finally, the fourth term on the RHS is identical to the second term except for that s in the denominator, which tells us that in time domain it is the time integral of the second term; $\int_0^t e^{\alpha t}dt = \frac{1}{\alpha}e^{\alpha t}|_0^t = \frac{1}{\alpha}(e^{\alpha t} - 1)$. Therefore, we can synthesize the total response as follows.

$$\begin{aligned} y(t) &= P_5(1+\alpha t)e^{\alpha t} + P_6 e^{\alpha t} + \eta A P_7 te^{\alpha t} + \eta A P_8\frac{1}{\alpha}(e^{\alpha t} - 1) \\ &= -\frac{\eta A P_8}{\alpha} + \left(\frac{\eta A P_8}{\alpha} + P_6 + P_5\right)e^{\alpha t} + (\eta A P_7 + P_5\alpha)te^{\alpha t} \end{aligned} \tag{3.64}$$

3.2.5 Simulation

Using the same value for damping b=700[Ns/cm], we determine k =2450[N/cm] using $b = 2\sqrt{km}$. Then, the two poles of the characteristic equation coincide at α=-7. The homogeneous, exogenous, and the total responses are illustrated in Fig.3.9

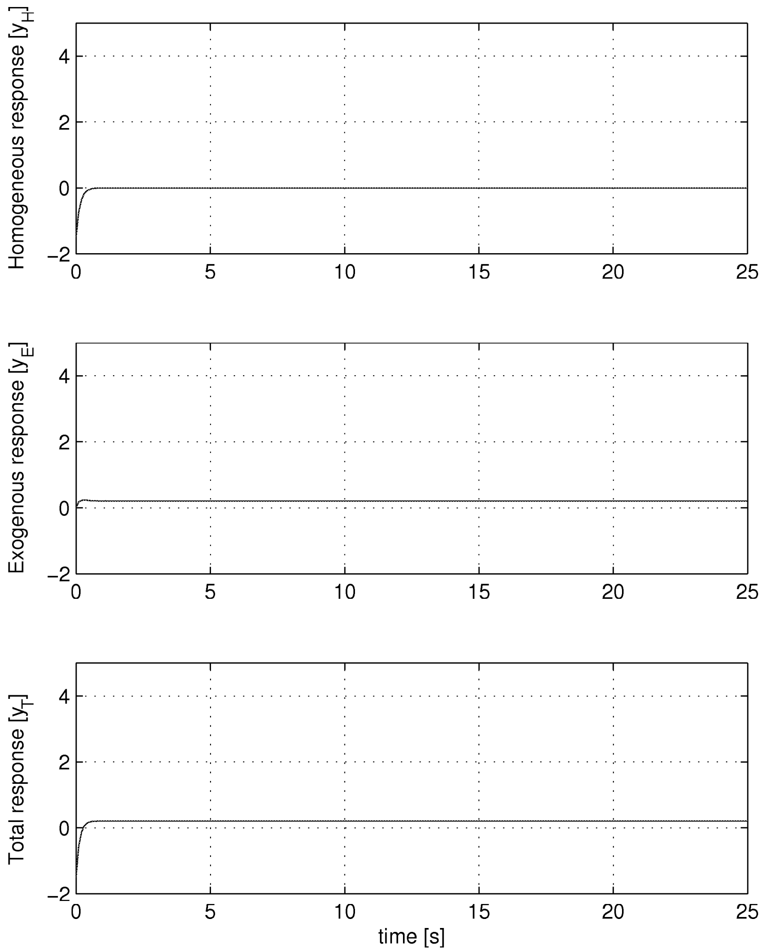


Figure 3.9: Shock absorber response when $b = 2\sqrt{km}$ (real, negative, and coincident poles)

We can compare this response with the one in Fig.3.8 in that stiffness was lower (k=125, so that the two poles were real and distinct), and understand

that the response stabilizes very quickly, with a little squeezing. This condition where the two poles are coincident is known as *critically damped.*
Other than comparing response, we may also want to take a closer look at the response then the system is critically damped as, which is illustrated in Fig.3.10

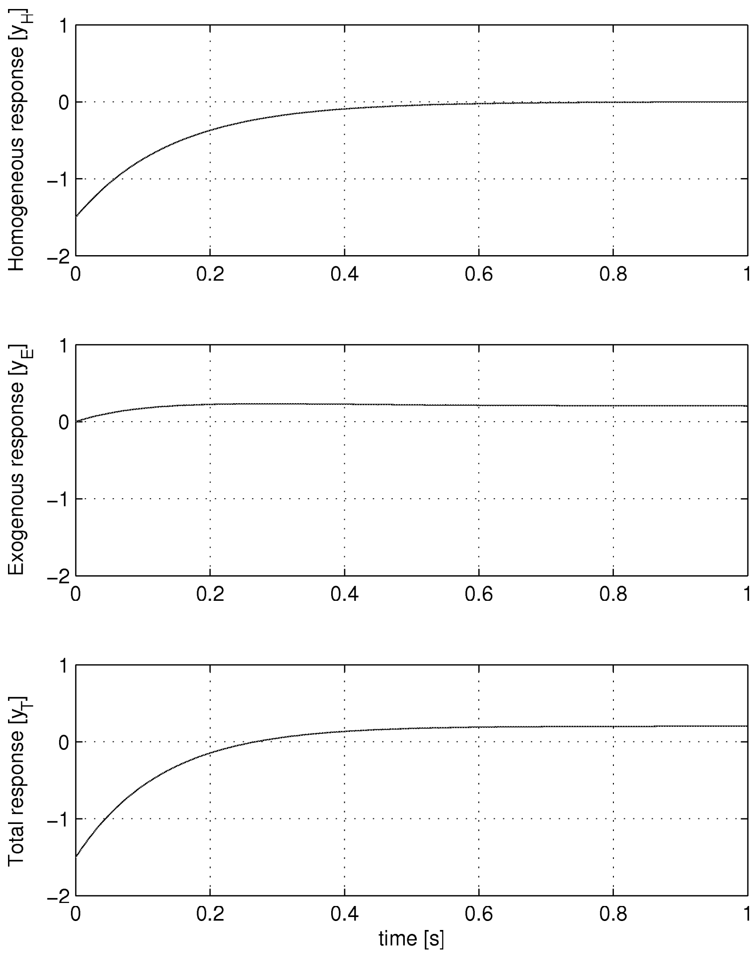


Figure 3.10: Closer look at the critically damped response of Fig.3.9

Case 3: $\sigma^2 - \rho < 0,\ b < 2\sqrt{km} \Rightarrow$ **complex conjugate pair of poles:**

In this case, the characteristic equation has a pair of complex conjugate poles

$$\alpha_1, \alpha_2 = -\sigma \pm j\omega \tag{3.65}$$

where $\omega = \sqrt{\rho - \sigma^2}$. Then, (3.52) can be written for a step input $F(s) = Au_s(t)$ as follows

$$\begin{aligned}
Y(s) &= \frac{y(0)s + [2\sigma y(0) + y'(0)]}{(s+\sigma-j\omega)(s+\sigma+j\omega)} + \frac{\eta}{(s+\sigma-j\omega)(s+\sigma+j\omega)}F(s) \\
&= \frac{K_1 s + K_2}{(s+\sigma)^2 - (j\omega)^2} + \frac{\eta}{(s+\sigma)^2 - (j\omega)^2}F(s) \\
&= K_1\frac{s}{(s+\sigma)^2+\omega^2} + K_2\frac{1}{(s+\sigma)^2+\omega^2} + \eta\frac{1}{(s+\sigma)^2+\omega^2}\frac{A}{s} \\
&= K_1\left[\frac{s+\sigma}{(s+\sigma)^2+\omega^2} - \frac{\sigma}{\omega}\frac{\omega}{(s+\sigma)^2+\omega^2}\right] + \frac{K_2}{\omega}\frac{\omega}{(s+\sigma)^2+\omega^2} \\
&\quad \frac{\eta}{\omega}\frac{\omega}{(s+\sigma)^2+\omega^2}\frac{A}{s}
\end{aligned} \tag{3.66}$$

The first term on the RHS clearly is a Laplace transform of a $\cos\omega t$ function as derived in (3.11), however, with s being replaced by $s+\sigma$, which is a result of an exponential scaling $e^{-\sigma t}$ as derived in (3.19). The second and third terms are Laplace transforms of $\sin\omega t$, but again exponentially scaled by $e^{-\sigma t}$. The fourth term looks like Laplace transform of an exponentially scalled function $e^{-\sigma t}\sin\omega t$, however, the s in the denominator tells us that it should be the integral of this function in $\int_0^t e^{-\sigma t}\sin\omega t dt$. This integral is evaluated in Appendix A. Therefore, we can synthesize the response $y(t)$ as follows.

$$\begin{aligned}
y(t) &= K_1 e^{-\sigma t}\cos\omega t - \frac{K_1\sigma}{\omega}e^{-\sigma t}\sin\omega t + \frac{K_2}{\omega}e^{-\sigma t}\sin\omega t \\
&\quad + \frac{\eta A}{\omega}\int_0^t e^{-\sigma t}\sin\omega t dt \\
&= e^{-\sigma t}\left\{K_1\cos\omega t + \frac{(K_2 - \sigma K_1)}{\omega}\sin\omega t\right\} \\
&\quad + \frac{\eta A}{\omega}\left\{\frac{\omega}{\omega^2+\sigma^2} - e^{-\sigma t}\sin(\omega t + \phi_E)\right\} \\
&= K e^{-\sigma t}\sin(\omega t + \phi_H) \\
&\quad + \frac{\eta A}{\omega}\left\{\frac{\omega}{\omega^2+\sigma^2} - e^{-\sigma t}\sin(\omega t + \phi_E)\right\}
\end{aligned} \tag{3.67}$$

where $K = \sqrt{K_1^2 + \frac{(K_2-\sigma K_1)^2}{\omega^2}}$, $\phi_H = \tan^{-1}\left(\frac{K_1\omega}{K_2-\sigma K_1}\right)$, and $\phi_E = \tan^{-1}\left(\frac{\omega}{\sigma}\right)$.

3.2.6 Simulation

By reducing damping $b = 300$ while keeping stiffness unchanged at $k = 2450$ characteristic equation assumes a pair of complex conjugate poles $\alpha_1, \alpha_2 = -3 \pm j6.3$. Under this condition, ϕ_H=-2.16[rad] ,and ϕ_E=0.32[rad]. Using the same initial conditions previously used, and for a 50[kg] person getting on board, the shock-absorber squeezes as illustrated in Fig.3.11.

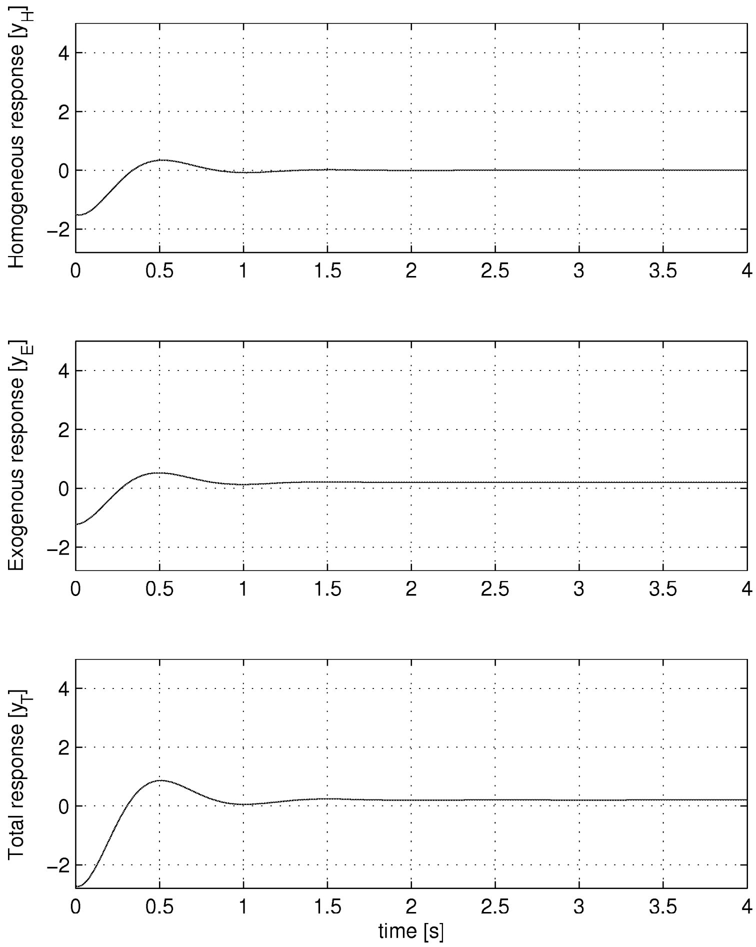


Figure 3.11: Shock-absorber response when it is under damped

where we can notice the decaying oscillatory behavior, which is called the **damped oscillation**. The imaginary part of the poles cause the oscillation whereas the -ve real part of the poles cause the decaying of the oscillation.

3.3 Transfer Function

If we cautiously observe the responses in Fig.3.7, Fig.3.8, Fig.3.9, Fig.3.10, and Fig.3.11 we witness the decaying nature of the homogeneous response, while the systems response gradually being over taken by the exogenous response. Therefore, it is justifiable that the homogeneous response is often dropped in modeling of systems. This is equivalent to assuming zero initial conditions $y(0) = y'(0).. = 0$, which is agreed upon by most practical systems that often stay at rest initially. Furthermore, in control system design, we are more interested in knowing how the exogenous input affects the response. Therefore, lets drop the homogeneous part of the RC circuit response in (3.47) and obtain the folowing relationship.

$$\begin{aligned} V(s) &= \frac{b}{s+a} V_s(s) \\ \frac{V(s)}{V_s(s)} &= \frac{b}{s+a} \\ G(s) &= \frac{b}{s+a} \end{aligned} \tag{3.68}$$

where $G(s)$ denotes the transfer function of the system, which tells how the exogenous response is created by the exogenous input. The system can now be illustrated by the block diagram in Fig.3.12.

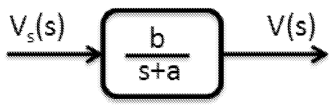


Figure 3.12: Transfer function block diagram of a first order system (RC circuit)

Similar way, we can reduce the shock absorber response in (3.52) and write the transfer function as follows

$$\begin{aligned} Y(s) &= \frac{\eta}{s^2 + 2\sigma s + \rho} F_s(s) \\ \frac{Y(s)}{F(s)} &= \frac{\eta}{s^2 + 2\sigma s + \rho} \\ G(s) &= \frac{\eta}{s^2 + 2\sigma s + \rho} \end{aligned} \tag{3.69}$$

The system can be illustrated by the following block diagram in Fig.3.13

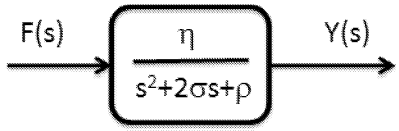


Figure 3.13: Transfer function block diagram of a second order system (shock absorber)

Example: Transfer Function of a Robot Link

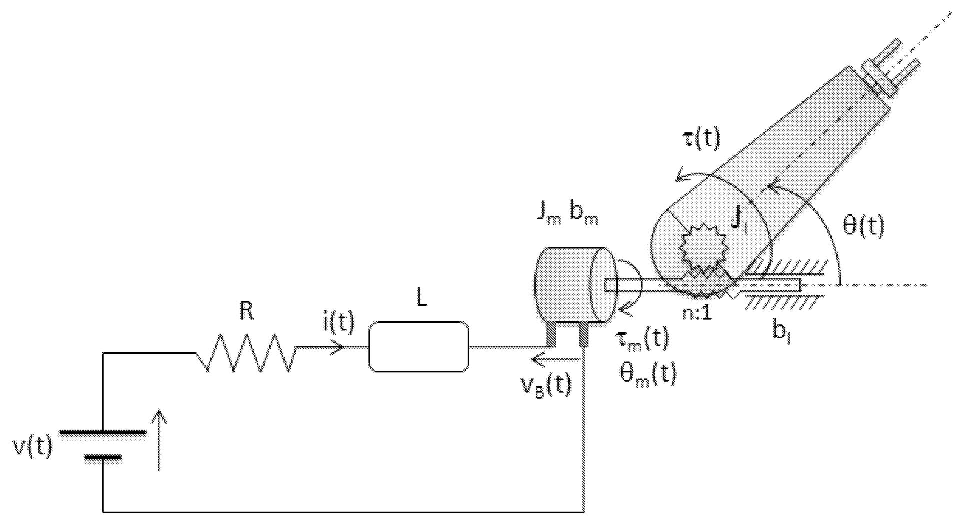


Figure 3.14: Robot arm and the joint motor

Figure 3.14 shows a robot arm and its joint motor, which is excited with a variable DC voltage $v(t)$. We need to find out how the arm position $\theta(t)$ changes when the voltage $v(t)$ changes, i.e., we need to determine the transfer function $\frac{\Theta(s)}{V(s)}$. Lets first introduce the variables and parameters listed up on the table 3.1.

For the electrical circuit we can write $v(t) = iR + L\frac{di(t)}{dt} + v_B(t)$. However, we know that $v_B(t) = k_B\dot{(\theta)}_m$. Therefore, by substitution $v(t) = iR + L\frac{di(t)}{dt} + k_B\dot{(\theta)}_m$, which in Laplace domain is

$$\begin{aligned} V(s) &= RI(s) + LsI(s) + k_B s\Theta_m(s) \\ &= [R + Ls]I(s) + k_B s\Theta_m(s) \end{aligned} \qquad (3.70)$$

total inertia on the motor shaft is the motor inertia plus the load inertia,

$v(t)$: armature voltage[V]	$i(t)$: armature current[A]
J_m: motor inertia [Kgm^2]	k_τ: torque constant [Nm/A]
n: gear ration	τ_l: load shaft torque [Kgm^2]
L: armature inductance [Vs/A]	$v_B(t)$: back electromotive force [V]
b_m: motor viscous damping constant [Nms/rad]	$\dot{\theta}_m(t)$: motor shaft speed [rad/s]
J_l: arm inertia [kgm^2]	R: armature resistance [Ω]
k_B: motor back emf constant [Vs/rad]	τ_m: motor shaft torque [Nm]
$\theta_m(t)$:motor shaft position [rad]	b_l: viscous damping constant of the arm [Nms/rad]
$\theta(t)$: arm position [rad]	

Table 3.1: Parameters and variables of the robot arm system

which is coupled through a $n:1$ reduction gear. Therefore, equivalent inertia on the motor shaft is $J_{eq} = J_m + \frac{1}{n^2}J_l$. Similarly, equivalent viscous friction coefficient on the motor shaft is $b_{eq} = b_m + \frac{1}{n^2}b_l$. Torque on the motor shaft overcomes the friction (velocity dependent), and drives the inertia (acceleration depends) as described by $\tau_m(t) = b_{eq}\dot{\theta}(t) + J_{eq}\ddot{\theta}_m(t)$. However, we know that $\tau_m(t) = k_\tau i(t)$. Therefore, by substitution $k_\tau i(t) = b_{eq}\dot{\theta}(t) + J_{eq}\ddot{\theta}_m(t)$, which in Laplace domain, gives the following expression for $I(s)$

$$\begin{aligned} I(s) &= \frac{1}{k_\tau}[J_{eq}s^2\Theta_m(s) + b_{eq}s\Theta_m(s)] \\ &= \frac{1}{k_\tau}[J_{eq}s + b_{eq}]s\Theta_m(s) \end{aligned} \tag{3.71}$$

By substitution from (3.71) to (3.70) for $I(s)$

$$\begin{aligned} V(s) &= \frac{1}{k_\tau}(R + Ls)(J_{eq}s + b_{eq})s\Theta_m(s) + k_B s\Theta_m(s) \\ k_\tau V(s) &= [(R + Ls)(J_{eq}s + b_{eq}) + k_\tau k_B]s\Theta_m(s) \\ \frac{\Theta_m(s)}{V(s)} &= \frac{k_\tau}{[(R + Ls)(J_{eq}s + b_{eq}) + k_\tau k_B]s} \end{aligned} \tag{3.72}$$

As we know $\theta_m(t) = n\theta(t)$, we can write the required transfer function as

$$\begin{aligned}\frac{\Theta_(s)}{V(s)} &= \frac{k_\tau/n}{[(R+Ls)(J_{eq}s+b_{eq})+k_\tau k_B s]s} \\ &= \frac{k_\tau/n}{LJ_{eq}s^3+(Lb_{eq}+RJ_{eq})s^2+(Rb_{eq}+k_Bk_\tau)s} \qquad (3.73)\end{aligned}$$

3.3.1 Experimental Determination of Transfer Function

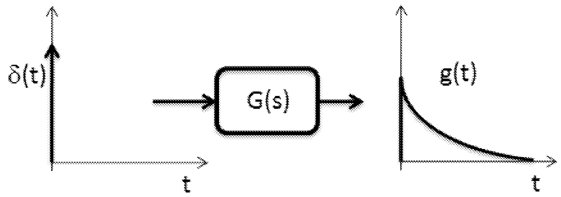


Figure 3.15: Unit impulse response of a plant

As derived in (3.45), exogenous response of the system $Y(s) = G(s)R(s)$. If we give a unit impulse input $r(t) = \delta(t) \Rightarrow R(s) = 1$ as shown in Fig.3.15, then

$$Y(s)|_{R(s)=1} = G(s) \qquad (3.74)$$

which says that the system transfer function is the Laplace transform of the unit impulse response. This provides a very useful experimental method to determine the transfer function of a system. A unit impulse $\delta(t)$ is given to the system and the response $g(t)$ is recorded as shown in Fig.3.15. The Laplace transform of this response is the transfer function of the system $G(s) = \mathcal{L}\{g(t)\}$.

3.4 Summary and Conclusion

Using Laplace transforms, differential equations (time domain) can be converted into algebraic equations in s-domain. These algebraic equations can be handled and manipulated much easier than dealing with their corresponding differential equations. We perform algebraic manipulations in s domain in such a way that we recognize familiar Laplace transforms, which help us to directly write down the solution in time-domain. In Matlab, systems can be easily constructed and their responses can be simulated quickly and effectively.

System's homogeneous response dies out with time, and most systems start with zero initial conditions. Therefore, we can reasonably drop the homogeneous response, and derive the relationship between exogenous input and response, which is known the transfer function of the system.