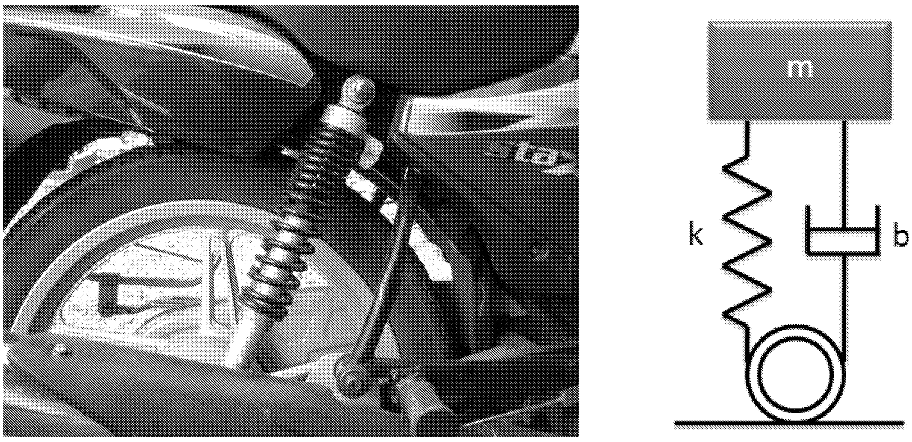
Chapter 2

System Model and Response

2.1 Mechanical System: Vehicle Shock-Absorber

Lets look at the motor cycle shock absorber mechanism shown in Fig.2.1, and try to understand its operation, and develop a mathematical model of it. We know by experience that shock-absorber absorbs vibrations generated by the irregularities on the road so that the rider does not feel such uncomfortable vibrations. The shock absorber has parallel arrangement of a spring k and a damper b connected between wheel and the seat as shown in Fig.2.1.



(a) (b)

Figure 2.1: (a) Motorcycle shock absorber, and (b) Equivalent damper spring mechanism

As shown in Fig.2.2, when the rider gets on board a downward force $f(t) = mg$ is applied on the shock-absorber. This force squeezes the spring-damper mechanism, and therefore makes the spring and damper generate opposing forces $ky(t)$ and $b\dot{y}(t)$ respectively, where k[N/m] is the spring constant, and b[N/ms^{-1}] is the damper coefficient. The free body diagram of the mass-spring-damper is shown in Fig.2.2(b).

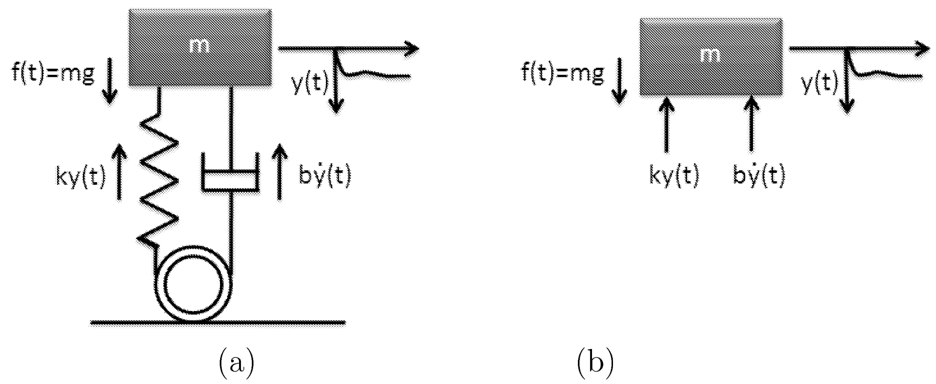


Figure 2.2: (a) Shock absorber forces (b), and (c) Free body diagram

According to the free body diagram there is an unbalanced upward force $f - ky(t) - b\dot{y}(t)$, which makes an upward acceleration $\ddot{y}(t)$ on the rider as given by

$$m\ddot{y}(t) = f - ky(t) - b\dot{y}(t) \tag{2.1}$$

Hence, the acceleration felt by the rider due to the bump on the road is given by the following differential equation

$$\ddot{y}(t) + \frac{b}{m}\dot{y}(t) + \frac{k}{m}y(t) = \frac{1}{m}f(t) \tag{2.2}$$

2.2 An Electrical System: RC Circuit

Lets look at the electrical circuit shown in Fig.2.3 and try to mathematically model the voltage across the capacitor. Basic components of electrical circuits are resisters ($v(t) = i(t)R$), capacitors ($i(t) = c\dot{v}(t)$), and inductors($v(t) = L\dot{i}(t)$). Lets consider the RC circuit shown below, and try to model the voltage $v(t)$ across the capacitor. We can now apply either Kirchoff's current law ($\sum_{node} i = 0$), which says that the total current entering a node is equal to the total current leaving the node. Or else we could use Kirchoff's voltage law ($\sum_{loop} v = 0$), which says that all voltages around a loop adds to zero. Lets apply the current law as follows

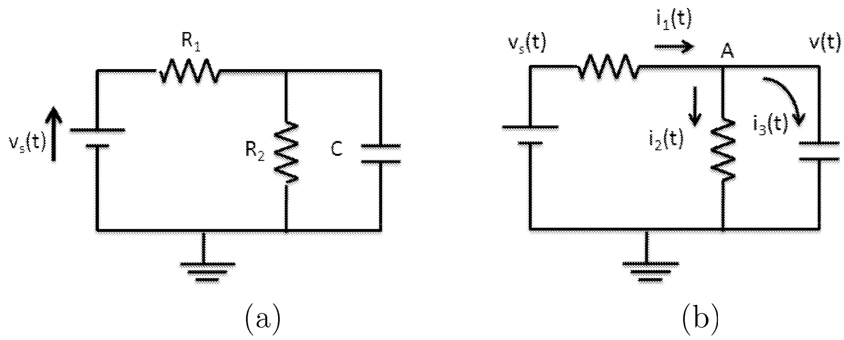


Figure 2.3: (a) Simple RC circuit and (b) currents at node A

$$\begin{aligned}
i_2(t) + i_3(t) &= i_1(t) \\
C\dot{v}(t) + \frac{1}{R_2}v(t) &= \frac{v_s(t) - v(t)}{R_1} \\
R_1R_2C\dot{v}(t) + R_1v(t) &= R_2(v_s(t) - v(t)) \\
R_1R_2C\dot{v}(t) + (R_1 + R_2)v(t) &= R_2v_s(t) \\
\dot{v}(t) + \frac{R_1 + R_2}{R_1R_2C}v(t) &= \frac{1}{R_1C}v_s(t) \\
\dot{v}(t) + av(t) &= bv_s(t) \qquad (2.3)
\end{aligned}$$

where $a = \frac{R_1+R_2}{R_1R_2C}$, and $b = \frac{1}{R_1C}$. This gives us the ordinary differential equation which describes how voltage across the capacitor changes in response of the supply voltage. However, we should not forget that we have assumed a constant capacitance C, which means that either $v_s(t)$ has to be DC source, or an AC supply with a very low frequency compared with charging and discharging of the capacitor.

A detailed account on electric and mechanical system modeling can be found in [4]. We can conclude that systems can be modeled by ordinary differential equations(ODE). Once we have derived the system's ODE we must solve it to get the system's response, i.e., $y(t)$ in the shock-absorber, and $v(t)$ in the RC circuit. Therefore, lets try to solve differential equations (2.2) and (2.3).

2.3 Response from Model Differential Equations

The response of a system is given by the solution of the systems ODE. There are two parts in the solution of an ODE, namely, (1) *initial condition response*, and (2) *steady state response*. Initial condition response is the part of the response that is due to the input ant output values at $t = 0$. Steady state response is the response when the system attains steady behavior.

2.3.1 First Order System (RC Circuit)

The response of first order RC circuit in (2.3) is the sun of *complementary function* $v_{CF}(t)$, and *particular integral* $v_{PI}(t)$ as follows

$$v(t) = v_{CF}(t) + v_{PI}(t) \tag{2.4}$$

The initial condition response $v_{ic}(t)$ is obtained by removing excitation function $v_s(t)$ from (2.3) as follows.

$$\dot{v}_{CF}(t) + av_{CF}(t) = 0 \tag{2.5}$$

Lets assume $v_{CF}(t) = Ae^{\alpha t}$ as a solution. If it is in fact a solution it must satisfy (2.5). Lets substitute $v_{CF}(t)$, and $\dot{v}_{CF}(t) = \alpha Ae^{\alpha t}$ onto (2.5) as follows.

$$\begin{aligned} \alpha Ae^{\alpha t} + aAe^{\alpha t} &= 0 \\ Ae^{\alpha t}(\alpha + a) &= 0 \end{aligned} \tag{2.6}$$

which is satisfied if $\alpha = -a$. Therefore, the initial condition solution is given by

$$v_{CF}(t) = Ae^{-at} \tag{2.7}$$

However, we still do not know the value of coefficient A, which we will find out later. The steady state response is obtained by removing response derivatives from (2.3) as follows

$$\begin{aligned} av_{PI}(t) &= bv_s(t) \\ v_{PI}(t) &= \frac{b}{a}v_s(t) \end{aligned} \tag{2.8}$$

Then, the total response is the sum of (2.7) and (2.8)

$$v(t) = Ae^{-at} + \frac{b}{a}v_s(t) \tag{2.9}$$

Now we can find an expression for coefficient A by writing (2.8) at $t = 0$ as follows.

$$\begin{aligned} v(0) &= A + \frac{b}{a}v_s(0) \\ A &= v(0) - \frac{b}{a}v_s(0) \end{aligned} \tag{2.10}$$

Therefore, the total response is

$$v(t) = \left\{v(0) - \frac{b}{a}v_s(0)\right\} e^{-at} + \frac{b}{a}v_s(t) \tag{2.11}$$

This shows that initial condition solution contains values of both $v(t)$ and $v_s(t)$ at $t = 0$, and it also shows that initial condition response decays with time. Total response can be solved for a known forcing function. Lets assume $v_s(t) = V_s$[V], a DC voltage source, for which the total response is as follows

$$\begin{aligned} v(t) &= \left\{v(0) - \frac{b}{a}V_s\right\} e^{-at} + \frac{b}{a}V_s \\ &= v(0)e^{-at} + \frac{b}{a}V_s\left\{1 - e^{-at}\right\} \end{aligned} \tag{2.12}$$

in which the first term on the right RHS can be identified as the *homogeneous response*(due to output itself) whereas the second term on the RHS can be identified as the *exogenous response* (due to external input)

2.3.2 Second Order System: Shock-Absorber

The response $y(t)$ of shock-absorber dynamics in (2.2) is the summation of the complementary function(homogeneous response) $y_{CF}(t)$, and the particular integral(exogeneous response) $y_{PI}(t)$

$$y(t) = y_{CF}(t) + y_{PI}(t) \tag{2.13}$$

Homogeneous Response (Complementary Function)

The homogeneous part of the system differential equation (2.2) is obtained by disregarding the external forcing function ($f(t) = 0$) as follows

$$\ddot{y}_{CF}(t) + 2\sigma\dot{y}_{CF}(t) + \rho y_{CF}(t) = 0 \tag{2.14}$$

where $2\sigma = b/m$, and $\rho = k/m$. We can immediately notice that this ODE is satisfied by $y_{CF}(t) = e^{\alpha t}$ and its derivatives $\dot{y}_{CF}(t) = A\alpha e^{\alpha t}$ and $\ddot{y}_{CF}(t) = A\alpha^2 e^{\alpha t}$. The corresponding value of the exponent α can be determined by substituting the candidate complementary function and its derivatives onto the homogeneous equation as follows

$$\begin{aligned} A\alpha^2 e^{\alpha t} + 2\sigma A\alpha e^{\alpha t} + \rho A e^{\alpha t} &= 0 \\ Ae^{\alpha t}(\alpha^2 + 2\sigma\alpha + \rho) &= 0 \\ \alpha^2 + 2\sigma\alpha + \rho &= 0 \end{aligned} \tag{2.15}$$

This quadratic equation is known as the characteristic equation of the system, as its solutions, which are called the poles of the system determines the characteristics of the system. We notice here that though we start with a single candidate complementary function, the homogeneous equation is satisfied by two such candidate functions affiliated with the two poles α_1, α_2 of the characteristic equation. It is also noticeable that there can be three different possible response corresponding to the nature of the poles being, real distinct, real coincident, or complex.

case 1: $\sigma^2 - \rho > 0$: In this case, $\alpha_1, \alpha_2 = -\sigma \pm \sqrt{\sigma^2 - \rho}$ are real(-ve) and distinct poles. The homogeneous response in this case is given by a general complementary function as follows

$$y_{CF}(t) = A_1 e^{(-\sigma + \sqrt{\sigma^2 - \rho})t} + A_2 e^{(-\sigma - \sqrt{\sigma^2 - \rho})t} \tag{2.16}$$

in that the second term has a greater negative exponent, thus, decays faster with time than the first term.

case 2: $\sigma^2 - \rho = 0$: In this case, the two poles are coincident $\alpha_1 = \alpha_2 = \alpha = -\sigma$. The candidate complementary function is obtained by time multiplication of one of the member solutions tested in the earlier case.

$$y_{CF}(t) = A_1 e^{-\sigma t} + A_2 t e^{-\sigma t} \tag{2.17}$$

case 3: $\sigma^2 - \rho < 0$: In this case, $\alpha_1, \alpha_2 = -\sigma \pm j\omega; j = \sqrt{-1}$ is a complex conjugate pair, where $\omega = \sqrt{\rho - \sigma^2}$. The two member solutions have a decay factor $e^{-\sigma t}$ and oscillatory factor $e^{\pm j\omega t}$, and in general, $\cos\omega t = (e^{j\omega t} + e^{-j\omega t})/2$ and $\sin\omega t = (e^{j\omega t} - e^{-j\omega t})/2j$ both qualify as solutions. Therefore, general homogeneous response is given by

$$\begin{aligned} y_{CF}(t) &= e^{-\sigma t}(P_1 \cos\omega t + P_2 \sin\omega t) \\ &= Ae^{-\sigma t}\cos(\omega t - \phi) \end{aligned} \tag{2.18}$$

Exogenous Response (Particular Integral)

Exogenous response is the solution of (2.2) due to the external forcing function alone. Obviously, the particular solution depends on the nature of the forcing function $f(t)$. It can be shown by differentiation and substitution onto (2.2) that following exogenous responses are valid for the given forcing functions

$$\begin{array}{rcl} f(t) & & y_{PI}(t) \\ t^n & \Rightarrow & a_0 + a_1 + \dot{+} a_n t^n \\ e^{at} & \Rightarrow & Ae^{at} \\ \cos\omega t & \Rightarrow & P_1 \cos\omega t + P_2 \sin\omega t \\ \sin\omega t & \Rightarrow & -do- \\ \cos\omega t + \sin\omega t & \Rightarrow & -do- \end{array} \tag{2.19}$$

Total Response

The total response (2.13) of the second order dynamics is assembled by adding homogeneous response (2.16) if roots of the characteristic equation are real, or (2.17) if roots are real and repeated, or (2.18) if roots is a complex conjugate pair, with the exogenous response that can be derived from the candidate solutions given in (2.19). The coefficients of these responses are to be determined using initial conditions $y(0)$, and $y'(0)$. This is the lengthy procedure of finding the system's total response

2.4 Linearization

A system is linear if it satisfies the theory of superposition, i.e., $f(x_1 + x_2) = f(x_1) + f(x_2)$. A nonlinear system (a function) does not satisfy superposition as illustrated in Fig.2.4. Many systems (electromechanical, hydraulic, pneumatic, etc.) involve nonlinear behaviors such as

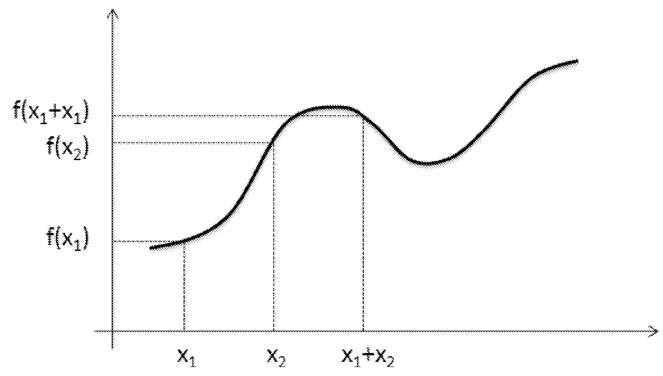


Figure 2.4: A nonlinear function $f(x_1 + x_2) \neq f(x_1) + f(x_2)$ does not satisfy the theory of superposition

- output may saturate for large inputs
- there may be dead zones where there is no output at all such as backlash of mechanical gears

Systems which are generally recognized as linear are in fact linear within the normal signal magnitudes (normal operation), however, under high values of signals they often show nonlinear behaviors. Most physical systems are inherently nonlinear by very nature. Nevertheless, we dislike to accept nonlinearities because it makes life very difficult for the control engineer. Therefore, most industrial systems are designed to operate within the close vicinity of a predetermined operating point around which plant model can be approximated with a linear model. Lets say $y = f(x)$ is a nonlinear function and we want to find out the linearized model of it around the operating point $(\bar{x})$. Lets use Taylor series and write the function around $\bar{x}$ as follows

$$\begin{aligned} y &= f(\bar{x}) + \frac{df}{dx}|_{\bar{x}}(x - \bar{x}) + \frac{1}{2!}\frac{d^2 f}{dx^2}|_{\bar{x}}(x - \bar{x})^2 + \dots \\ y^* &= f(\bar{x}) + \frac{df}{dx}|_{\bar{x}}(x - \bar{x}) \end{aligned} \quad (2.20)$$

where y^* is the approximated linear model of the plant. The approximation is justifies because of the fact that all higher order terms of $(x - \bar{x} = \delta x)$ getting increasingly smaller.

2.4.1 Example

Problem: Use the nonlinear function $z = xy$, and determine the linear approximation around the operating point $(\bar{x}, \bar{y}) = (6, 11)$. If the plant operates within the dynamic range of $5 \leq x \leq 7$, and $10 \leq y \leq 12$ determine the error of approximation when the plant operates at $(7, 12)$.

Solution: We can determine partial derivatives of z as $\frac{\partial z}{\partial x} = y$, and $\frac{\partial z}{\partial y} = x$. From (2.20),

$$\begin{aligned} z^* &= z|_{(6,11)} + \frac{\partial z}{\partial x}|_{(6,11)}(x-6) + \frac{\partial z}{\partial y}|_{(6,11)}(y-11) \\ &= 6 \times 11 + y|_{(6,11)}(x-6) + x|_{(6,11)}(y-11) \\ &= 66 + 11(x-6) + 6(y-11) \\ &= 11x + 6y - 66 \end{aligned} \tag{2.21}$$

The error e of approximation at $(7, 12)$ is

$$\begin{aligned} e|_{(7,12)} &= z|_{(7,12)} - z^*|_{(7,12)} \\ &= xy|_{(7,12)} - (11x + 6y - 66)|_{(7,12)} \\ &= 7 \times 12 - (11 \times 7 + 6 \times 12 - 66) \\ &= 84 - (77 - 72 - 66) \\ &= 1 \end{aligned} \tag{2.22}$$

The percentage error is $e|_{(7,12)}\% = \frac{e|_{(7,12)}}{z^*|_{(6,11)}} \times 100\% = \frac{1}{84} \times 100\% = 1.19\%$.

2.5 Summary and Conclusion

We use laws of physics such as Newton's laws and Kirchoff's laws, and derive the model of dynamic systems. The modeling process almost always result in an ordinary differential equation. The model differential equation can be solved to determine system's response. The solution has two parts; one describing how the response change by itself, and the other part shows how the response change due to exogenous inputs. To solve differential equations can be troublesome, specially when the system order is high. Therefore mathematical tools that can solve differential equations conveniently are needed. The next chapter on Laplace transforms will describe such methods using Laplace transforms.